

A Note on Normal Forms of Quantum States and Separability

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Abstract

We study the normal form of multipartite density matrices. It is shown that the correlation matrix (CM) separability criterion can be improved from the normal form we obtained under filtering transformations. Based on CM criterion the entanglement witness is further constructed in terms of local orthogonal observables for both bipartite and multipartite systems.

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One of the important problems in the theory of quantum entanglement is the separability: to decide whether or not a given quantum state is entangled. A multipartite state $\rho_{AB\dots C}$ is called separable if it can be written as $\rho_{AB\dots C} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \dots \otimes \rho_i^C$, where $\rho_i^A, \rho_i^B, \dots, \rho_i^C$ are density matrices on subsystems A, B, \dots, C , and $p_i \geq 0, \sum_i p_i = 1$. There have been many separability criteria such as Bell inequalities [1], PPT (positive partial transposition) [2], entanglement witnesses [3, 4], realignment [5], local uncertainty relations [6, 7] etc. In [8] by using the Bloch representation of density matrices the author has presented a separability criterion, which is further generalized to multipartite case [9]. In [10] the normal form of a bipartite state has been obtained. We indicate that this normal form can be used to improve the separability criteria from Bloch representation and local uncertainty relations. In this note we study the normal form of multipartite density matrices. We show that the correlation matrix (CM) criterion can be improved from the normal form we obtained under filtering transformations. Based on CM criterion we further construct the entanglement witness in terms of local orthogonal observables (LOOs) [11] for both bipartite and multipartite systems.

For bipartite case, $\rho \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim \mathcal{H}_A = M, \dim \mathcal{H}_B = N, M \leq N$, is mapped to the following form under local filtering transformations [12]:

$$\rho \rightarrow \tilde{\rho} = \frac{(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger}{\text{Tr}[(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger]}, \quad (1)$$

where $F_{A/B} \in GL(M/N, \mathbb{C})$ are arbitrary invertible matrices. This transformation is also known as stochastic local operations assisted by classical communication (SLOCC). By the definition it is obvious that filtering transformation will preserve the separability of a quantum state.

It has been shown that under local filtering operations one can transform a strictly positive ρ into a normal form [10],

$$\tilde{\rho} = \frac{(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger}{\text{Tr}[(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger]} = \frac{1}{MN} (I + \sum_{i=1}^{M^2-1} \xi_i G_i^A \otimes G_i^B), \quad (2)$$

where $\xi_i \geq 0$, G_i^A and G_i^B are some traceless orthogonal observables. The matrices F_A and F_B can be obtained by minimizing the function

$$f(A, B) = \frac{\text{Tr}[\rho(A \otimes B)]}{(\det A)^{1/M}(\det B)^{1/N}}, \quad (3)$$

where $A = F_A^\dagger F_A$ and $B = F_B^\dagger F_B$. In fact, one can choose $F_A^0 \equiv |\det(\rho_A)|^{1/2M}(\sqrt{\rho_A})^{-1}$, and $F_B^0 \equiv |\det(\rho_B')|^{1/2N}(\sqrt{\rho_B'})^{-1}$, where $\rho_B' = \text{Tr}_A(I \otimes (\sqrt{\rho_A})^{-1} \rho I \otimes (\sqrt{\rho_A})^{-1})$. Then by the iteration one can get the optimal A and B. In particular, there is a matlab code available in [13]. The normal form of a product state (if exists) must be proportional to identity.

For bipartite separable states ρ , the CM separability criterion [8] says that

$$\|T\|_{KF} \leq \sqrt{MN(M-1)(N-1)}, \quad (4)$$

where T is an $(M^2 - 1) \times (N^2 - 1)$ matrix with $T_{ij} = MN \cdot \text{Tr}(\rho \lambda_i^A \otimes \lambda_j^B)$, $\|T\|_{KF}$ stands for the trace norm of T , $\lambda_k^{A/B}$ s are the generators of $SU(M/N)$ and have been chosen to be normalized, $\text{Tr} \lambda_k^{(A/B)} \lambda_l^{(A/B)} = \delta_{kl}$.

As the filtering transformation does not change the separability of a state, one can study the separability of $\tilde{\rho}$ instead of ρ . Under the normal form (2) the criterion (4) becomes

$$\sum_i \xi_i \leq \sqrt{MN(M-1)(N-1)}. \quad (5)$$

In [6] a separability criterion based on local uncertainty relation (LUR) has been obtained. It says that for any separable state ρ ,

$$1 - \sum_k \langle G_k^A \otimes G_k^B \rangle - \frac{1}{2} \langle G_k^A \otimes I - I \otimes G_k^B \rangle^2 \geq 0, \quad (6)$$

where $G_k^{A/B}$ s are LOOs such as the normalized generators of $SU(M/N)$ and $G_k^A = 0$ for $k = M^2 + 1, \dots, N^2$. The criterion is shown to be strictly stronger than the realignment criterion [5]. Under the normal form (2) criterion (6) becomes

$$\begin{aligned} 1 &= \sum_k \langle G_k^A \otimes G_k^B \rangle - \frac{1}{2} \langle G_k^A \otimes I - I \otimes G_k^B \rangle^2 \\ &= 1 - \frac{1}{\sqrt{MN}} - \frac{1}{MN} \sum_k \xi_k - \frac{1}{2} (\sum_k \langle G_k^A \rangle^2 + \sum_k \langle G_k^B \rangle^2 - 2 \sum_k \langle G_k^A \rangle \langle G_k^B \rangle) \\ &= 1 - \frac{1}{MN} \sum_k \xi_k - \frac{1}{2} \left(\frac{1}{M} + \frac{1}{N} \right) \geq 0, \end{aligned}$$

i.e.

$$\sum_k \xi_k \leq MN - \frac{M+N}{2}. \quad (7)$$

As $\sqrt{MN(M-1)(N-1)} \leq MN - \frac{M+N}{2}$ holds for any M and N , from (5) and (7) it is obvious that the CM criterion recognizes entanglement better when the normal form is taken into account.

We now consider multipartite systems. Let ρ be a strictly positive density matrix in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, $\dim \mathcal{H}_i = d_i$. ρ can be generally expressed in terms of the $SU(n)$ generators λ_{α_k} [9],

$$\begin{aligned} \rho = & \frac{1}{\prod_i^N d_i} \left(\otimes_j^N I_{d_j} + \sum_{\{\mu_1\}} \sum_{\alpha_1} \mathcal{T}_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_1}^{\{\mu_1\}} + \sum_{\{\mu_1 \mu_2\}} \sum_{\alpha_1 \alpha_2} \mathcal{T}_{\alpha_1 \alpha_2}^{\{\mu_1 \mu_2\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \right. \\ & + \sum_{\{\mu_1 \mu_2 \mu_3\}} \sum_{\alpha_1 \alpha_2 \alpha_3} \mathcal{T}_{\alpha_1 \alpha_2 \alpha_3}^{\{\mu_1 \mu_2 \mu_3\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \lambda_{\alpha_3}^{\{\mu_3\}} \\ & + \cdots + \sum_{\{\mu_1 \mu_2 \cdots \mu_M\}} \sum_{\alpha_1 \alpha_2 \cdots \alpha_M} \mathcal{T}_{\alpha_1 \alpha_2 \cdots \alpha_M}^{\{\mu_1 \mu_2 \cdots \mu_M\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \cdots \lambda_{\alpha_M}^{\{\mu_M\}} \\ & \left. + \cdots + \sum_{\alpha_1 \alpha_2 \cdots \alpha_N} \mathcal{T}_{\alpha_1 \alpha_2 \cdots \alpha_N}^{\{1, 2, \cdots, N\}} \lambda_{\alpha_1}^{\{1\}} \lambda_{\alpha_2}^{\{2\}} \cdots \lambda_{\alpha_N}^{\{N\}} \right), \end{aligned} \quad (8)$$

where $\lambda_{\alpha_k}^{\{\mu_k\}} = I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes \lambda_{\alpha_k} \otimes I_{d_{\mu_k+1}} \otimes \cdots \otimes I_{d_N}$ with λ_{α_k} appears at the μ_k th position and

$$\mathcal{T}_{\alpha_1 \alpha_2 \cdots \alpha_M}^{\{\mu_1 \mu_2 \cdots \mu_M\}} = \frac{\prod_{i=1}^M d_{\mu_i}}{2^M} \text{Tr}[\rho \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \cdots \lambda_{\alpha_M}^{\{\mu_M\}}].$$

The generalized CM criterion says that: if ρ in (8) is fully separable, then

$$\|\mathcal{T}^{\{\mu_1, \mu_2, \cdots, \mu_M\}}\|_{KF} \leq \sqrt{\frac{1}{2^M} \prod_{k=1}^M d_{\mu_k} (d_{\mu_k} - 1)}, \quad (9)$$

for $2 \leq M \leq N$, $\{\mu_1, \mu_2, \cdots, \mu_M\} \subset \{1, 2, \cdots, N\}$. The KF norm is defined by

$$\|\mathcal{T}^{\{\mu_1, \mu_2, \cdots, \mu_M\}}\|_{KF} = \max_{m=1, 2, \cdots, M} \|\mathcal{T}_{(m)}\|_{KF},$$

where $\mathcal{T}_{(m)}$ is a kind of matrix unfolding of $\mathcal{T}^{\{\mu_1, \mu_2, \cdots, \mu_M\}}$.

The criterion (9) can be improved by investigating the normal form of (8).

[Theorem 1] By filtering transformations of the form

$$\tilde{\rho} = F_1 \otimes F_2 \otimes \cdots \otimes F_N \rho F_1^\dagger \otimes F_2^\dagger \otimes F_N^\dagger, \quad (10)$$

where $F_i \in GL(d_i, \mathbb{C})$, $i = 1, 2, \cdots, N$, followed by normalization, any strictly positive state ρ can be transformed into a normal form

$$\begin{aligned} \rho = & \frac{1}{\prod_i^N d_i} \left(\otimes_j^N I_{d_j} + \sum_{\{\mu_1 \mu_2\}} \sum_{\alpha_1 \alpha_2} \mathcal{T}_{\alpha_1 \alpha_2}^{\{\mu_1 \mu_2\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} + \sum_{\{\mu_1 \mu_2 \mu_3\}} \sum_{\alpha_1 \alpha_2 \alpha_3} \mathcal{T}_{\alpha_1 \alpha_2 \alpha_3}^{\{\mu_1 \mu_2 \mu_3\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \lambda_{\alpha_3}^{\{\mu_3\}} \right. \\ & + \cdots + \sum_{\{\mu_1 \mu_2 \cdots \mu_M\}} \sum_{\alpha_1 \alpha_2 \cdots \alpha_M} \mathcal{T}_{\alpha_1 \alpha_2 \cdots \alpha_M}^{\{\mu_1 \mu_2 \cdots \mu_M\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \cdots \lambda_{\alpha_M}^{\{\mu_M\}} \\ & \left. + \cdots + \sum_{\alpha_1 \alpha_2 \cdots \alpha_N} \mathcal{T}_{\alpha_1 \alpha_2 \cdots \alpha_N}^{\{1, 2, \cdots, N\}} \lambda_{\alpha_1}^{\{1\}} \lambda_{\alpha_2}^{\{2\}} \cdots \lambda_{\alpha_N}^{\{N\}} \right). \end{aligned} \quad (11)$$

[Proof] Let D_1, D_2, \cdots, D_N be the sets of density matrices of the N subsystems. The cartesian product $D_1 \times D_2 \times \cdots \times D_N$ consisting of all product density matrices $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N$ with

normalization $\text{Tr}\rho_i = 1$, $i = 1, 2, \dots, N$, is a compact set of matrices on the full Hilbert space \mathcal{H} . For the given density matrix ρ we define the following function of ρ_i

$$f(\rho_1, \rho_2, \dots, \rho_N) = \frac{\text{Tr}[\rho(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N)]}{\prod_{i=1}^N \det(\rho_i)^{1/d_i}}.$$

The function is well-defined on the interior of $D_1 \times D_2 \times \dots \times D_N$ where $\det \rho_i > 0$. As ρ is assumed to be strictly positive, we have $\text{Tr}[\rho(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N)] > 0$. Since $D_1 \times D_2 \times \dots \times D_N$ is compact, we have $\text{Tr}[\rho(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N)] \geq C > 0$ with a lower bound C depending on ρ .

It follows that $f \rightarrow \infty$ on the boundary of $D_1 \times D_2 \times \dots \times D_N$ where at least one of the ρ_i s satisfies that $\det \rho_i = 0$. It follows further that f has a positive minimum on the interior of $D_1 \times D_2 \times \dots \times D_N$ with the minimum value attained for at least one product density matrix $\tau_1 \otimes \tau_2 \otimes \dots \otimes \tau_N$ with $\det \tau_i > 0$, $i = 1, 2, \dots, N$. Any positive density matrix τ_i with $\det \tau_i > 0$ can be factorized in terms of Hermitian matrices F_i as

$$\tau_i = F_i^\dagger F_i \quad (12)$$

where $F_i \in GL(d_i, \mathbb{C})$. Denote $F = F_1 \otimes F_2 \otimes \dots \otimes F_N$, so that $\tau_1 \otimes \tau_2 \otimes \dots \otimes \tau_N = F^\dagger F$. Set $\tilde{\rho} = F\rho F^\dagger$ and define

$$\begin{aligned} \tilde{f}(\rho_1, \rho_2, \dots, \rho_N) &= \frac{\text{Tr}[\tilde{\rho}(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N)]}{\prod_{i=1}^N \det(\rho_i)^{1/d_i}} \\ &= \prod_{i=1}^N \det(\tau_i)^{1/d_i} \cdot \frac{\text{Tr}[\rho(F_1^\dagger \rho_1 F_1 \otimes F_2^\dagger \rho_2 F_2 \otimes \dots \otimes F_N^\dagger \rho_N F_N)]}{\prod_{i=1}^N \det(\tau_i)^{1/d_i} \det(\rho_i)^{1/d_i}} \\ &= \prod_{i=1}^N \det(\tau_i)^{1/d_i} \cdot f(F_1^\dagger \rho_1 F_1, F_2^\dagger \rho_2 F_2, \dots, F_N^\dagger \rho_N F_N). \end{aligned}$$

We see that when $F_i^\dagger \rho_i F_i = \tau_i$, \tilde{f} has a minimum and

$$\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N = (F^\dagger)^{-1} \tau_1 \otimes \tau_2 \otimes \dots \otimes \tau_N F^{-1} = I.$$

Since \tilde{f} is stationary under infinitesimal variations about the minimum it follows that

$$\text{Tr}[\tilde{\rho} \delta(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N)] = 0$$

for all infinitesimal variations,

$$\begin{aligned} \delta(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N) &= \delta\rho_1 \otimes I_{d_2} \otimes \dots \otimes I_{d_N} + I_{d_1} \otimes \delta\rho_2 \otimes I_{d_3} \otimes \dots \otimes I_{d_N} \\ &\quad + \dots + I_{d_1} \otimes I_{d_2} \otimes \dots \otimes I_{d_{N-1}} \otimes \delta\rho_N, \end{aligned}$$

subjected to the constraint $\det(I_{d_i} + \delta\rho_i) = 1$, which is equivalent to $\text{Tr}(\delta\rho_i) = 0$, $i = 1, 2, \dots, N$, using $\det(e^A) = e^{\text{Tr}A}$ for a given matrix A . Thus, $\delta\rho_i$ can be represented by the SU generators, $\delta\rho_i = \sum_k \delta c_k^i \lambda_k^i$. It follows that $\text{Tr}(\tilde{\rho} \lambda_{\alpha_k}^{\{\mu_k\}}) = 0$ for any α_k and μ_k . Hence the terms proportional to $\lambda_{\alpha_k}^{\{\mu_k\}}$ in (8) disappear. \square

[Corollary] The normal form of a product state in \mathcal{H} must be proportional to the identity.

[Proof] Let ρ be such a state. From (11), we get that

$$\tilde{\rho}_i = \text{Tr}_{1,2,\dots,i-1,i+1,\dots,N} \rho = \frac{1}{d_i} I_{d_i}. \quad (13)$$

Therefore for a product state ρ we have

$$\rho = \rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_N = \frac{1}{\prod_{i=1}^N d_i} \otimes_{i=1}^N I_{d_i}.$$

□

To show the separability of multipartite states in terms of their normal forms (11) we consider the PPT entangled edge state [14]

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (14)$$

mixed with noises:

$$\rho_p = p\rho + \frac{(1-p)}{8} I_8.$$

Select $a = 2, b = 3$, and $c = 0.6$. Using the criterion in [9] we get that ρ_p is entangled for $0.92744 < p \leq 1$. But after transforming ρ_p to its normal form (11), the criterion can detect entanglement for $0.90285 < p \leq 1$.

Here we indicate that the filtering transformation does not change the PPT property. Let $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ be PPT, i.e. $\rho^{T_A} \geq 0$, and $\rho^{T_B} \geq 0$. Let $\tilde{\rho}$ be the normal form of ρ . From (1) we have

$$\tilde{\rho}^{T_A} = \frac{(F_A^* \otimes F_B) \rho^{T_A} (F_A^T \otimes F_B^\dagger)}{\text{Tr}[(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger]}.$$

For any vector $|\psi\rangle$, we have

$$\langle \psi | \tilde{\rho}^{T_A} | \psi \rangle = \frac{\langle \psi | (F_A^* \otimes F_B) \rho^{T_A} (F_A^T \otimes F_B^\dagger) | \psi \rangle}{\text{Tr}[(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger]} \equiv \langle \psi' | \rho^{T_A} | \psi' \rangle \geq 0,$$

where $|\psi'\rangle = \frac{(F_A^T \otimes F_B^\dagger) |\psi\rangle}{\sqrt{\text{Tr}[(F_A \otimes F_B) \rho (F_A \otimes F_B)^\dagger]}}$. $\tilde{\rho}^{T_B} \geq 0$ can be proved similarly. This property is also valid for multipartite case. Hence a bound entangled state will be bound entangled under filtering transformations.

For N-partite systems in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_N$ ($N \geq 2$) with $\dim \mathcal{H}_i = d_i$, $i = 1, 2, \dots, N$, the local orthogonal observables (LOOs) can be given in the following way. Assume $d_n \equiv \max\{d_i, i = 1, 2, \dots, N\}$. One can choose d^2 observables G_k^n associated with the subsystem \mathcal{H}_n . For other

subsystems with smaller dimensions, say \mathcal{H}_1 , one can choose d_1^2 observables G_k^1 , $k = 1, 2, \dots, d_1^2$ and set $G_k^1 = 0$ for $k = d_1^2 + 1, \dots, d^2$. Based on CM criterion we can further construct entanglement witness (EW) in terms of such LOOs. EW [11] is an observable of the composite system that has (i) nonnegative expectation values in all separable states and (ii) at least one negative eigenvalue (or equivalently, can recognize at least one entangled state).

We first consider bipartite systems in $\mathcal{H}_A^M \otimes \mathcal{H}_B^N$ with $M \leq N$.

[Theorem 2] For any LOOs G_k^A and G_k^B ,

$$W = I - \alpha \sum_{k=0}^{N^2-1} G_k^A \otimes G_k^B$$

is an EW, where $\alpha = \frac{\sqrt{MN}}{\sqrt{(M-1)(N-1)+1}}$ and

$$G_0^A = \frac{1}{\sqrt{M}} I_M, \quad G_0^B = \frac{1}{\sqrt{N}} I_N. \quad (15)$$

[Proof] Let $\rho = \sum_{l,m=0}^{N^2-1} T_{lm} \lambda_l^A \otimes \lambda_m^B$ be a separable state, where $\lambda_k^{A/B}$ are normalized generators of $SU(M/N)$ with $\lambda_0^A = \frac{1}{\sqrt{M}} I_M$, $\lambda_0^B = \frac{1}{\sqrt{N}} I_N$. Any other LOOs $G_k^{A/B}$ fulfill (15) can be obtained from these λ s through orthogonal transformations $\mathcal{O}^{A/B}$, $G_k^{A/B} = \sum_{l=0}^{N^2-1} \mathcal{O}_{kl}^{A/B} \lambda_l$, where $\mathcal{O}^{A/B} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}^{A/B} \end{pmatrix}$, $\mathcal{R}^{A/B}$ are $(N^2-1) \times (N^2-1)$ orthogonal matrices. We have

$$\begin{aligned} \text{Tr} \rho W &= 1 - \alpha \frac{1}{\sqrt{MN}} - \alpha \sum_{k=1}^{N^2-1} \sum_{l,m=1}^{N^2-1} \mathcal{R}_{kl}^A \mathcal{R}_{km}^B \text{Tr} \rho (\lambda_l^A \otimes \lambda_m^B) \\ &= \frac{\sqrt{(M-1)(N-1)}}{\sqrt{(M-1)(N-1)+1}} - \frac{1}{\sqrt{MN}(\sqrt{(M-1)(N-1)+1})} \sum_{k=1}^{N^2-1} \sum_{l,m=1}^{N^2-1} \mathcal{R}_{kl}^A T_{lm} \mathcal{R}_{km}^B \\ &= \frac{\sqrt{(M-1)(N-1)}}{\sqrt{(M-1)(N-1)+1}} - \frac{1}{\sqrt{MN}(\sqrt{(M-1)(N-1)+1})} \text{Tr}(\mathcal{R}^A T (\mathcal{R}^B)^T) \\ &\geq \frac{\sqrt{MN(M-1)(N-1)} - \|T\|_{KF}}{\sqrt{MN}(\sqrt{(M-1)(N-1)+1})} \geq 0, \end{aligned}$$

where we have used $\text{Tr}(\mathcal{R}T) \leq \|T\|_{KF}$ for any unitary \mathcal{R} in the first inequality and the CM criterion in the second inequality.

Now let $\rho = \frac{1}{MN} (I_{MN} + \sum_{i=1}^{M^2-1} s_i \lambda_i^A \otimes I_N + \sum_{j=1}^{N^2-1} r_j I_M \otimes \lambda_j^B + \sum_{i=1}^{M^2-1} \sum_{j=1}^{N^2-1} T_{ij} \lambda_i^A \otimes \lambda_j^B)$ be a state in $\mathcal{H}_A^M \otimes \mathcal{H}_B^N$ which violates the CM criterion. Denote $\sigma_k(T)$ the singular values of T . By singular value decomposition, one has $T = U^\dagger \Lambda V^*$, where Λ is a diagonal matrix with $\Lambda_{kk} = \sigma_k(T)$. Now choose LOOs to be $G_k^A = \sum_l U_{kl} \lambda_l^A$, $G_k^B = \sum_m V_{km} \lambda_m^B$ for $k = 1, 2, \dots, N^2-1$ and $G_0^A = \frac{1}{M} I_M, G_0^B = \frac{1}{N} I_N$. We obtain

$$\text{Tr} \rho W = 1 - \alpha \frac{1}{\sqrt{MN}} - \alpha \sum_{k=1}^{N^2-1} \sum_{l,m=1}^{N^2-1} U_{kl} V_{km} \text{Tr} \rho (\lambda_l^A \otimes \lambda_m^B)$$

$$\begin{aligned}
&= \frac{\sqrt{(M-1)(N-1)}}{\sqrt{(M-1)(N-1)}+1} - \frac{1}{\sqrt{MN}(\sqrt{(M-1)(N-1)}+1)} \text{Tr}(UTV^T) \\
&= \frac{\sqrt{MN(M-1)(N-1)} - \|T\|_{KF}}{\sqrt{MN}(\sqrt{(M-1)(N-1)}+1)} < 0
\end{aligned}$$

where the CM criterion has been used in the last step. \square

As the CM criterion can be generalized to multipartite form in [9], we can also define entanglement witness for multipartite system in $\mathcal{H}_1^{d_1} \otimes \mathcal{H}_2^{d_2} \otimes \dots \otimes \mathcal{H}_N^{d_N}$. Set $d(M) = \max\{d_{\mu_i}, i = 1, 2, \dots, M\}$. Choose LOOs $G_k^{\{\mu_i\}}$ for $0 \leq k \leq d(M)^2 - 1$ with $G_0^{\{\mu_i\}} = \frac{1}{d_{\mu_i}} I_{d_{\mu_i}}$ and define

$$W^{(M)} = I - \beta^{(M)} \sum_{k=0}^{d(M)^2-1} G_k^{\{\mu_1\}} \otimes G_k^{\{\mu_2\}} \otimes \dots \otimes G_k^{\{\mu_M\}}, \quad (16)$$

where $\beta^{(M)} = \frac{\sqrt{\prod_{i=1}^M d_{\mu_i}}}{1 + \sqrt{\prod_{i=1}^M (d_{\mu_i} - 1)}}$, $2 \leq M \leq N$. One can prove that (16) is an EW candidate for multipartite states. First we assume $\|\mathcal{T}^{(M)}\|_{KF} = \|\mathcal{T}_{(m_0)}\|_{KF}$. Note that for any $\mathcal{T}_{(m_0)}$, there must exist an elementary transformation P such that $\sum_{k=1}^{d(M)^2-1} \mathcal{T}_{kk\dots k}^{\{\mu_1\mu_2\dots\mu_M\}} = \text{Tr}(\mathcal{T}_{(m_0)}P)$. Then for an N-partite separable state we have

$$\begin{aligned}
\text{Tr} \rho W^{(M)} &= 1 - \beta^{(M)} \frac{1}{\sqrt{\prod_{i=1}^M d_{\mu_i}}} - \beta^{(M)} \frac{1}{\prod_{i=1}^M d_{\mu_i}} \sum_{k=1}^{d(M)^2-1} \mathcal{T}_{kk\dots k}^{\{\mu_1\mu_2\dots\mu_M\}} \\
&= 1 - \beta^{(M)} \frac{1}{\sqrt{\prod_{i=1}^M d_{\mu_i}}} - \beta^{(M)} \frac{1}{\prod_{i=1}^M d_{\mu_i}} \text{Tr}(\mathcal{T}_{(m_0)}P) \\
&\geq 1 - \beta^{(M)} \frac{1}{\sqrt{\prod_{i=1}^M d_{\mu_i}}} - \beta^{(M)} \frac{1}{\prod_{i=1}^M d_{\mu_i}} \|\mathcal{T}_{(m_0)}\|_{KF} \\
&\geq 1 - \beta^{(M)} \frac{1}{\sqrt{\prod_{i=1}^M d_{\mu_i}}} - \beta^{(M)} \frac{1}{\prod_{i=1}^M d_{\mu_i}} \sqrt{\prod_{k=1}^M d_{\mu_k} (d_{\mu_k} - 1)} \\
&= 0
\end{aligned}$$

for any $2 \leq M \leq N$, where we have used that P must be orthogonal matrix and $\text{Tr}(MU) \leq \|M\|_{KF}$ for any unitary U at the first inequality. The second inequality is due to the generalized CM criterion.

By choosing proper LOOs it is also easy to show that $W^{(M)}$ has negative eigenvalues. For example for three qubits case, taking the normalized pauli matrices as LOOs, one find a negative eigenvalue of $W^{(M)}$, $\frac{1-\sqrt{3}}{2}$.

We have studied the normal form of multipartite density matrices. It has been shown that separability criteria can be improved by transforming the states to their normal forms through filtering transformations. The entanglement witness has been constructed in terms of local orthogonal observables for both bipartite and multipartite systems. Here we considered only the strictly positive (full rank) states. Although full rank is a sufficient condition for the existence of

the normal forms, in fact for many rank deficiency density matrices their normal forms can be also calculated.

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